Solitons in anharmonic chains with negative group velocity

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(Received 16 May 2007; revised manuscript received 27 July 2007; published 6 December 2007)

We consider the problem of the soliton dynamics in anharmonic chains with nearest-neighbor interactions and negative group velocity. By using the quasicontinuum approach we derive analytic expressions for envelope solitons. We observe that these solitons are stable under collisions and, moreover, in the absence of losses they can propagate with nearly any desired velocity. Energy loss effects are discussed by considering the presence of Stokes and hydrodynamical damping. Numerical simulations are performed showing a good agreement with the theory.

DOI: 10.1103/PhysRevE.76.066602 PACS number(s): 05.45.Yv, 05.40.−a, 63.10.+a

Phenomena associated with wave-packet propagation in a medium with negative group velocity (NGV) is often counterintuitive. For instance, backward pulse propagation through a NGV medium [1] was recently experimentally observed. Similar behavior has been theoretically predicted for soliton propagation in left-handed surfaces [2,3]. Other phenomena such as superluminal group velocity has been experimentally observed in erbium-doped optical fibers [1] and coaxial photonic crystals [4], and predicted for metamaterials [5].

Nonlinearity combined with NGV give rise to interesting phenomena in wave propagation. A complete understanding of these phenomena is important in a view of possible applications. So, some physical insight can be gained by analyzing analytically the propagation of solitary waves (in short solitons) in a NGV medium. In fact, one of the most simple systems to study effects due to the interplay between nonlinearity and NGV is the finite chain of electrical coupled nonlinear oscillators, as shown in Fig. 1 [6–8]. Notice that anharmonic chains play an important role in physics and the soliton concept is useful in explaining essential features, such as energy transport. Moreover, the study of anharmonic chains in the context of metamaterials and special materials is even stronger when looking into the possibility of actually synthesizing the metamaterial by resembling the anharmonic chain response [9].

In this work, we shall present a theory for the soliton propagation on the simple NGV anharmonic chain, as shown in Fig. 1 [6–8]. We note that the existence of breathers [10] and solitons in chains of split-ring oscillators has been predicted in the frame of the nonlinear Schrödinger equation [11].

In Fig. 1 we show a chain of identical electrical oscillators with anomalous dispersion. Here we consider that either capacitances or inductances are nonlinear. Notice, for instance, that varactor diodes can be used as nonlinear capacitances, $C=\mu_0 W(\Delta v_n)$, which are functions of voltage differences Δv_n [6,12]. On the other hand, ferrimagnetic materials have been suggested for obtaining a nonlinear inductive response, $L=\mu_0 W(I_n)$, function of a current I_n [13]. μ_0 is constant and the nonlinear function $W(u_n)=u_n+\gamma u_n^p$ with $p>1$ and γ is constant. Notice that u_n can be identified in Fig. 1 as either a potential difference $\Delta v_n = v_n - v_{n-1}$ across a nonlinear capacitance or a current I_n flowing through a nonlinear inductance.

In the following we consider the cases of constant inductances, $L=\kappa_0$, and nonlinear capacitances, $C=\mu_0 W(u_n)$, and vice versa [$C=\kappa_0$ and $L=\mu_0 W(u_n)$].

The Lagrangian of the NGV anharmonic chain (Fig. 1) reads as

$$\mathbb{L} = \sum_{n\in\mathbb{Z}} \left[\frac{\mu_0}{2}\left(\frac{d\Delta q_n}{dt}\right)^2 + \frac{\mu_0\gamma}{p+1}\left(\frac{d\Delta q_n}{dt}\right)^{p+1} - \frac{q_n^2}{2\kappa_0} \right], \quad (1)$$

where $\Delta q_n = q_n - q_{n-1}$ and the generalized coordinate

$$q_n = \begin{cases} \displaystyle\int v_n dt & \text{for } L=\kappa_0, \quad C=\mu_0 W(\Delta v_n), \\ \displaystyle\sum_j^n \int I_j dt & \text{for } C=\kappa_0, \quad L=\mu_0 W(I_n). \end{cases} \quad (2)$$

By using Lagrangian formulation it is straightforward to obtain equations of motion

$$\alpha \frac{d^2}{dt^2}[W(u_{n-1}) - 2W(u_n) + W(u_{n+1})] = u_n + G, \quad (3)$$

where $u_n = d\Delta q_n/dt$ and $\alpha=\mu_0\kappa_0$ is constant. The term G in Eq. (3) has been added to represent the energy losses of the system. If nothing else is said we can consider $G=0$. Notice that Eq. (3) can also be obtained by using Kirchhoff's circuit laws.

In order to study the soliton solutions of Eq. (3) we consider the case $p=3$ for the nonlinear function $W(u_n)$. For the more general case of cubic plus quartic anharmonic term, the calculations are similar [14], yet require a greater technical effort [14–16].

By examining the harmonic case of Eq. (3), i.e., $\gamma=0$, in the absence of losses it is straightforward to determine the dispersion relation ω_h and the group velocity v_g,

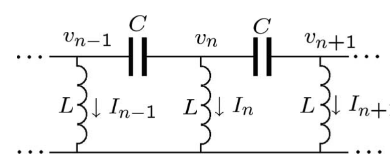

FIG. 1. Left-handed anharmonic chain of identical oscillators: $v_n - v_{n-1}$ is the potential difference across the nth capacitance C and I_n is the current through the nth inductance L.

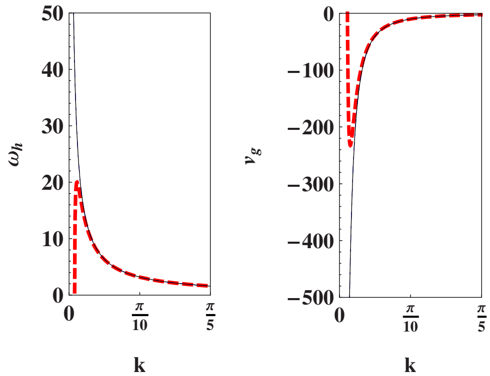

FIG. 2. (Color online) Dispersion relation ω_h vs k (left-hand side), and group velocity v_h vs k (right-hand side). Solid lines: Curves in the absence of damping ($\nu_S = \nu_h = 0$) or in the presence of the hydrodynamical damping ($\nu_S = 0$ and $\nu_h = 0.05$). Note that curves are undistinguishable. Dashed lines: Curves in the presence of Stokes damping ($\nu_S = 0.05$ and $\nu_h = 0$), $\alpha = 1$.

$$\omega_h = \frac{1}{2\sqrt{\alpha}}\left|\csc\left(\frac{k}{2}\right)\right|, \quad v_g = \frac{-1}{4\sqrt{\alpha}}\csc\left(\frac{k}{2}\right)\cot\left(\frac{k}{2}\right). \quad (4)$$

For $k \to 0$ both ω_g and v_g diverge, i.e., $\omega_h \to \infty$ and $v_g \to -\infty$ (see Fig. 2). On the other hand, $v_g \to 0$ for $k \to \pi$.

Since Eq. (3) in the absence of losses ($G=0$) cannot be solved analytically, we use the quasicontinuum approximation (QCA) to calculate soliton solutions [16–20] which we use as initial conditions for the numerical simulations. In order to proceed with the QCA, Eq. (3) with $G=0$ is transformed into an operator equation using the full Taylor expansion of the potential term $[W(u_{n\pm1}) \to \exp(\pm\partial_n)W(u(x,t))]$

$$4\alpha \sinh^2(\partial_n/2)\partial_t^2 W(u(n,t)) = u(n,t), \quad (5)$$

where n can be regarded as a continuous variable.

Notice that Eq. (5), which is still exact, cannot bear pulse soliton solutions. This can be easily seen if we integrate Eq. (5) twice with respect to time. In this case on the right-hand side of Eq. (5) appears a term $\sim\int_0^t dt' \int_0^{t'} dt'' u(n,t'')$, which diverges for $t \to +\infty$ if $u(n,t)$ is a single pulse.

Here we proceed to use the QCA for oscillatory excitations, as was performed for a conventional anharmonic chain of oscillators in Refs. [14,16]. So, here we assume that there exist an oscillatory excitation whose envelope moves with a constant velocity v and therefore depends on the coordinate $z = n - vt$. In Ref. [14] an ansatz is given by an expansion into harmonics $\theta = kn - \omega t + \delta$,

$$u_n(t) = \sum_{m\in Z} \chi_m(z)e^{im\theta}, \quad \chi_m = \tilde{\chi}_{-m}, \quad (6)$$

where k is the wave number, δ is phase, and ω is a frequency to be determined later. Inserting Eq. (6) in Eq. (5) and Fourier transforming the resultant equation we obtain

$$\alpha\tilde{W}_m(q) = a_m(q)\tilde{\chi}_m(q) \quad (7)$$

with

$$a_m(q) = \frac{1}{4\sin^2\left(\frac{mk+q}{2}\right)(m\omega + vq)^2}. \quad (8)$$

Here, the tilde marks the Fourier transformed function, e.g.,

$$\tilde{W}_m(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dz e^{iqz} W_m(z), \quad (9)$$

where W_m is an abbreviation which collects all products of envelope functions that belong to the same harmonic $e^{im\theta}$, i.e.,

$$\sum_m W_m(z)e^{im\theta} = W\left(\sum_l \chi_l(z)e^{il\theta}\right). \quad (10)$$

It is remarkable to note in Eq. (8) that in contrast with the conventional anharmonic chain the singularity for $q \to 0$ remains if $m=0$, i.e., Eq. (3) cannot bear pulse solitons, as mentioned above. On the other hand, it is also remarkable to note that for $m \neq 0$ and $q \to 0$ the wave number k can take infinitely small values since the frequency term $\omega \sim \omega_h$, as we show below.

The QCA consists in a formal solution of Eq. (7) for $\tilde{W}_m(q)$ and in an expansion of the fraction $a_m(q)$ [Eq. (8)] for small q [14,16], i.e., $a_m(q) \simeq \Sigma_n a_{nm} q^n$. If we consider the expansion of the fraction $a_m(q)$ up to second order in q, forcing the first-order term of the expansion to be zero, and transforming back to the position space, we find a second-order differential equation for the first harmonic χ_1, namely,

$$a_{01}\chi_1 - a_{21}\partial_z^2\chi_1 = \alpha(\chi_1 + 3\gamma|\chi_1|^{p-1}\chi_1), \quad p=3. \quad (11)$$

Notice that the Vakhitov-Kolokov stability criterion for an elliptic equation, as Eq. (11), shows that for $p<4$ soliton solutions exist, i.e., they do not blow up [21,22].

Integrating by parts Eq. (11) and integrating again leads to a solution for the envelope which depends on the sign of γ,

$$u(z) = \sqrt{\alpha\frac{2(1-v_0^2)}{3\gamma v_0^2}}\,\mathrm{sech}\left(2\sqrt{\frac{1-v_0^2}{8\omega_h^2-1}}z\right)\cos(\theta) \quad (12)$$

for $\gamma>0$ and $v_0^2<1$, while for $\gamma<0$ and $v_0^2>1$ the dark envelope soliton reads as

$$u(z) = \sqrt{\alpha\frac{(1-v_0^2)}{3\gamma v_0^2}}\tanh\left(\sqrt{\frac{2(v_0^2-1)}{8\omega_h^2-1}}z\right)\cos(\theta). \quad (13)$$

In Eqs. (12) and (13) $\omega = v_0\omega_h$ and the soliton velocity is $v = v_0 v_g$. Since ω_h and v_g depend on k [see Eq. (4)], it is convenient to choose k and v_0 as free parameters [14].

As we mentioned above, ω here turns out to be proportional to ω_h, so the singularity in Eq. (8) for $q \to 0$ and $m \neq 0$ is removed, i.e., small-amplitude solitons exist for any finite k value in the first Brillouin zone.

It is also interesting to note that since the soliton velocity v is proportional to the group velocity $v_g \in [0, -\infty)$, the soliton can in principle move with nearly any desired velocity.

Notice also in Eqs. (12) and (13) that for a given v_0 the soliton widths are proportional to ω_h. In fact, for $k \to 0$ ($v_g \to -\infty$) we obtain that $k\omega_h \to 1/\sqrt{\alpha}$. This indicates that for

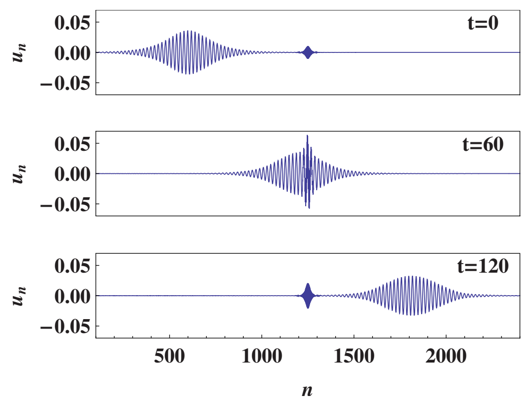

FIG. 3. (Color online) Collision between a bright soliton ($k=\pi/10$, $v_0=-0.999$) and a bright breather ($k=\pi$, $v_0=0.999$): snapshots for $t=0$, 60, and 120.

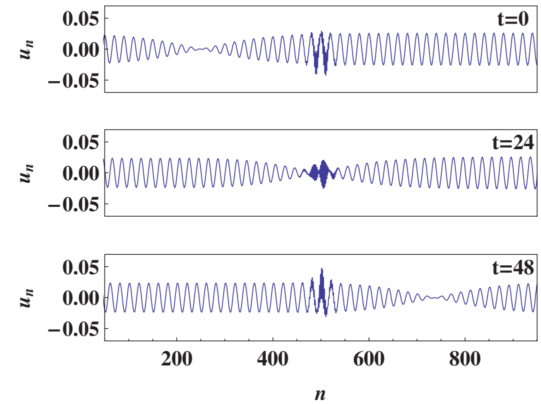

FIG. 4. (Color online) Collision between a dark soliton ($k=\pi/10$, $v_0=-1.001$) and a bright breather ($k=\pi$, $v_0=0.999$): snapshots for $t=0$, 24, and 48.

small values of k (high soliton velocities) the number of oscillations of the carrier wave being modulated by the envelope remains constant, i.e., the soliton width scales with the wavelength of the carrier wave. So, for $k\to 0$ the wavelength and the soliton width tend to infinity.

Here, it is worth mentioning that, although the soliton solutions exist, modulational instability (MI) can be present in the system. In fact, by using modulational stability analysis in discrete lattices in the fashion similar to Ref. [23] we observe that MI can appear for harmonic solutions even when $\gamma<0$. MI is easily observed in numerical simulations for narrow solitons and breather solutions, as mentioned below.

In order to simulate Eq. (3) we always choose $\alpha=|\gamma|=1$. At $t=0$ the chain is initialized with the soliton shapes given in Eqs. (12) and (13). The simulations have been performed for chains with at least 1000 lattice points. The time integration is carried out by using the Heun method [25] which has been successfully used for the numerical solution of nonlinear lattice systems [15,16,20,24].

Since the system is a NGV medium, the parameter v_0 must be chosen negative (positive) in order that the solitons move to the right-hand (left-hand) direction. On the other hand, $|v_0|\simeq 1$ in order to get small soliton amplitudes and wide soliton widths. For large amplitudes and narrow widths MI is observed, i.e., in the bright-soliton case, the initial pulse envelope gets broader as time goes on and eventually breaks in a series of wave packets.

For $k=\pi$ we observe a breather solution, i.e., an envelope with zero velocity whose amplitude oscillates rapidly in time. For long time scales ($t>1000$) MI is observed in the same fashion as for narrow solitons. We note that MI in the case of dark breathers ($\gamma<0$) is observed as broadening of the dark envelope and appearance of wave packets nearby.

For $k<\pi$ we observe soliton solutions which are stable under collisions, i.e., the soliton envelope is not altered after the collision. In Figs. 3 and 4 we show, for example, collisions of bright and dark solitons, respectively, with a bright breather. In particular in Fig. 3 it is also possible to observe the amplitude oscillations in time of the bright breather, as mentioned above.

We note that in the case of dark-dark collisions it is difficult to observe collisions for solitons with different k, since the carrier wave of one dark soliton hides the shape of the other, and vice versa. For dark solitons with the same k we observe stable collisions, but their amplitude oscillates in time.

We have performed simulations up to $k=\pi/100$ which corresponds to soliton velocities around 1000. In Fig. 5 we show, for example, the position of a bright soliton vs time for three different velocities obtained from numerical simulations and compared with the expected theoretical result. A very good agreement is observed. For larger velocities (smaller k) the simulations are difficult since the soliton width gets larger and larger, and the step size of the numerical integrator must be chosen smaller and smaller, due to the high soliton velocity.

So far, we have considered the soliton propagation without energy dissipation ($G=0$). But in order to consider a more realistic system we have included energy loss effects due to the elements of the system. In this case we have considered that every capacitance and inductance has a single resistance R_C and R_L connected in parallel, respectively. In this case the function G in Eq. (3) reads as

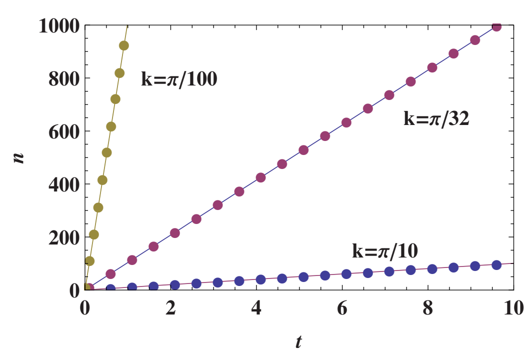

FIG. 5. (Color online) Position (n) of a bright soliton vs time (t) for $v_0=-0.999$ and $k=\pi/10$ ($v=10.09$), $k=\pi/32$ ($v=103.71$), and $k=\pi/100$ ($v=1013.17$). Dotted lines are results from simulations and solid lines are results from the theory [$n=vt$, where $v=v_0v_g$ with v_g given by Eq. (4)].

$$G = \nu_S \frac{du_n}{dt} - \nu_h\left(\frac{du_{n+1}}{dt} - 2\frac{du_n}{dt} + \frac{du_{n-1}}{dt}\right), \qquad (14)$$

where $\nu_S = L/R_L$ and $\nu_h = L/R_C$ are the Stokes and the hydrodynamical damping coefficients, respectively. We note that Eq. (14) is exact for nonlinear capacitances, while it is an approximation for nonlinear inductances. In fact, for the case of nonlinear inductances the damping terms are nonlinear, however, since we are considering small amplitude solitons, Eq. (14) turns out to be a good approximation for this case.

Damping effects on lattice solitons in anharmonic chains have been extensively studied [15,16,20,24], showing an exponential decay of the amplitude in time. An exponential coefficient multiplying the soliton amplitude can be estimated by solving the harmonic system, i.e., Eqs. (3) and (14) with $\gamma=0$, for a wave packet. It is straightforward to determine that the soliton amplitude decays like $u \sim e^{-\nu_s \omega_h^2 t/2}$ for the Stokes damping ($\nu_S \neq 0$ and $\nu_h = 0$) and like $u \sim e^{-\nu_h t/(2\alpha)}$ for the hydrodynamical damping ($\nu_h \neq 0$ and $\nu_S = 0$). These estimations fit very well results from numerical simulations, showing that the Stokes-damping effect is much stronger than the hydrodynamical one. In fact, we observe that in the presence of the Stokes damping the soliton amplitude $u \to 0$ for $k \to 0$. It is also easy to check, by looking at the dispersion relation in the presence of the Stokes damping (see Fig. 2), that the group velocity vanishes for $k \to 0$. So, no soliton propagation is possible for infinitesimal small k values when the Stokes damping is present. In contrast, the hydrodynamical-damping effect on the group velocity dispersion is negligible for $k \to 0$ (see Fig. 2: the curves in the presence of hydrodynamical damping or in the total absence of damping are undistinguishable).

In order to observe soliton propagation in a realistic experimental realization the losses must be reduced, in particular those related with the inductance since they are responsible for the Stokes damping. Another possibility consists of adding energy to the system by, for example, connecting to every node of the chain a current source controlled by the node voltage v_n. In fact, to some extent, a similar idea has been used in pulse propagation in nonlinear optical systems with NGV whose losses have been compensated by a gain in the system [1].

In conclusion, we have shown soliton solutions for the most simple NGV anharmonic chain of electrical oscillators. We have shown that this anharmonic chain cannot bear pulse solitons, but envelope solitons which are stable under collisions for finite wave numbers in the first Brillouin zone. We observe that these envelope solitons in the absence of losses can propagate with nearly any desired velocity, i.e., $v \in [0, \infty)$. We have also noted that for small wave-number values the soliton width scales with the wavelength of the carrier wave. Modulational instability has been observed for narrow solitons as well as for breather solutions. Finally, we have included energy losses by considering the presence of both the Stokes and the hydrodynamical damping. We have observed that in particular Stokes damping, which is related to losses in the inductances, affects strongly the soliton dynamics for small wave numbers.

Despite the fact that the experimental realization of this NGV anharmonic chain may be difficult, it is interesting to observe that soliton dynamics presented here, to some extent, resemble phenomena already observed in nonlinear optics such as superluminal and backward propagation.

It also is worth mentioning that by using the continuum approximation the basic discrete system, given in Eq. (3), can be approached by several NGV versions of very important differential equations, namely the nonlinear Schrödinger equation, the Boussinesq equation, and the Korteweg-de Vries equation. Since solutions obtained with the quasicontinuum approximation used here usually are more general than those obtained with the continuum approximation, it can be expected that reductions of the soliton solutions given here may also lead to solutions of NGV versions of the systems mentioned above. Finally, the present work is not only of relevance for metamaterial applications, but also can be of interest in nonlinear laser physics, Bose-Einstein condensate physics, and hydrodynamics.

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